The goal of this note is to present an upper bound on $\alpha(G_1 \times \cdots \times G_m)$ which depends only on $\alpha(G_1), \cdots, \alpha(G_m)$. In the next section we shall present the best possible such upper bound and show that the bound remains sharp when the factors G_i are all isomorphic. This is the case of importance for the study of zero-error probability codes.

3. Main Results

THEOREM 1

$\alpha(G_1 \times \cdots \times G_m) \leq M(\alpha_1 + 1, \alpha_2 + 1, \cdots, \alpha_m + 1) - 1$ *where* $\alpha_i = \alpha(G_i)$.

Proof: Let X be a subset of $G_1 \times \cdots \times G_m$ of size $\alpha(G_1 \times \cdots \times G_m)$ such that no two vertices of X are connected in $G_1 \times \cdots \times G_m$. Thus, if $x = (x_1, \cdots, x_m)$ and $y = (y_1, \cdots, y_m)$ are vertices from X, there will be at least one index i for which $x_i \neq y_i$ and x_i and y_i are not connected in G_i. Let us color the complete graph whose vertex set is X as follows: the edge joining x and y is colored with the ith color if i is the *smallest* index for which $x_i \neq y_i$ and x_i and y_i are not connected in G_i. Now if $\alpha(G_1 \times \cdots \times G_m) \geq M(\alpha_1 + 1, \cdots, \alpha_m + 1)$, then by Ramsey's theorem, we conclude that for some index i, there exists a complete $(\alpha_i + 1)$-subgraph of the ith color. But this would imply the existence of $\alpha_i + 1$ vertices in G_i, no two of which were connected, in conflict with the definition of $\alpha(G_i)$. Theorem 1 is proved.

THEOREM 2. Given integers $\alpha_1, \alpha_2, \cdots, \alpha_m$, there exist graphs $G_1, G_2, \cdots, G_m$ such that $\alpha(G_i) = \alpha_i$ and $\alpha(G_1 \times \cdots \times G_m) = M(\alpha_1 + 1, \cdots, \alpha_m + 1) - 1$.

Proof. From the definition of the Ramsey Number $M(\alpha_1 + 1, \cdots, \alpha_m + 1) = M$ we know that there is a coloring of the complete $M - 1$ graph in m colors such that the largest complete i-colored subgraph is on α_i nodes, for all i. For each i, let G_i be the graph on these $M - 1$ vertices which includes all edges which are, in fact, not colored with the ith color; thus $\alpha(G_i) = \alpha_i$. But it is easy to see that in $G_1 \times \cdots \times G_m$, the diagonal [i.e., the set of vertices of the form $(x, x, \cdots, x)$] is completely disconnected; if the edge connecting x and y is of the ith color, then $(x, x, \cdots, x)$ and $(y, y, \cdots, y)$ are not connected in the ith component and so not connected in $G_1 \times \cdots \times G_m$. Thus $\alpha(G_1 \times \cdots \times G_m) \geq M - 1$. But Theorem 1 forces $\alpha(G_1 \times \cdots \times G_m) \leq M - 1$, and so equality holds. This proves Theorem 2.

THEOREM 3. Given integers α and m, there exists a graph G with $\alpha(G) = \alpha$ and $\alpha(G^m) = M(\alpha + 1, \cdots, \alpha + 1) - 1$.

Proof. We begin with the graphs G_i as constructed in the proof of Theorem 2. Let G be the graph which consists of one copy each of $G_1, G_2, \ldots, G_m$, with every vertex from G_i connected to every vertex in G_j if $i \neq j$. Then clearly $\alpha(G) = \alpha$. Also, $G \times G \times \cdots \times G$ contains a subgraph isomorphic to $G_1 \times \cdots \times G_m$, and so $\alpha(G^m) \geq \alpha(G_1 \times \cdots \times G_m) = M - 1$. Again, $\alpha(G^m) = M - 1$ follows from Theorem 1. Theorem 3 is proved.

4. Discussion

While in one sense these theorems completely settle the question of giving bounds for $\alpha(G_1 \times \cdots \times G_m)$ in terms of the $\alpha(G_i)$, in another sense much is left to be desired, since most of the Ramsey Numbers are unknown. Thus, the only known nontrivial Ramsey Numbers are

$M(3,3) = 6,\ M(3,4) = 9,\ M(3,5) = 14,\ M(3,6) = 18,$
$M(3,7) = 23$

$M(4,4) = 18$

$M(3,3,3) = 17$

However, there are a variety of weak upper bounds known, the easiest to work with being $M(\alpha, \beta) \leq \binom{\alpha+\beta-2}{\alpha-1}$.

Reference

1. Hall, M. Jr., *Combinatorial Theory*, Blaisdell Publishing Co., New York, 1967.

D. Combinatorial Communication: An Upper Bound on the Free Distance of a Tree Code,

J. Layland and R. McEliece

1. Introduction

In SPS 37-50, Vol. III, pp. 248–252, McEliece and Rumsey obtained an upper bound on the free distance of a systematic convolutional code of rate $1/v$, where v is a positive integer. In this note we will modify that argument slightly and obtain an upper bound on the free distance of an arbitrary tree code of finite memory, systematic or not. The bounds obtained are exactly the same as those for convolutional codes with the same parameters. These upper bounds provide strong evidence that nonsystematic codes have considerably larger free distance than systematic codes. This is one reason why nonsystematic codes will be used in upcoming projects. In *Subsection 3*, we point out that the so-called *Griesmer bound* for linear codes can sometimes be used to tighten these bounds for convolutional codes. Finally in *Subsection 4* we present a short table of the bounds for particular values of the parameters.

2. The Bound

A rate k/n, constraint length K (memory $K-1$) binary tree code can be described as follows: If the information stream is partitioned into blocks of k bits each, say $I = (I_1^{(1)}, I_1^{(2)}, \cdots, I_1^{(k)}, I_2^{(1)}, I_2^{(2)}, \cdots, I_2^{(k)}, I_3^{(1)}, \cdots)$, then the output of the encoder is a sequence of blocks of n bits each, $C = (C_1^{(1)}, C_1^{(2)}, \cdots, C_1^{(n)}, C_2^{(1)}, C_2^{(2)}, \cdots, C_2^{(n)}, C_3^{(1)}, \cdots)$.

Here $C_i^{(j)}$ depends only on the information symbols $I_{i-s}^{(t)}$ for $s = 0, 1, \cdots, K-1$, $t = 1, 2, \cdots, k$. The free distance of this code is defined as the minimum Hamming distance between two of the (infinite) code words C.

Let T represent the set of all code words in the tree code, and T_h represent those code words corresponding to information streams which have $I_i^{(j)} = 0$ for $i > h$. Clearly $d_{\min}(T) \leqq d_{\min}(T_h)$ for all $h \geqq 1$. On the other hand T_h can be thought of as a block code of length $n(h+K-1)$, since for $i \geqq h+K$, $C_j^{(j)}$ depends only on I_l^m for $l \geqq h+1$, and these information bits are all zero.

Now according to Plotkin's bound (Ref. 1, Theorem 13.49), a code of length n with M code words has minimum distance $\leqq n/(2(1/1 - M^{-1}))$. Since T_h has block length $n(h+K-1)$ and 2^{hk} code words, we arrive at

THEOREM 1

$$d_{\text{free}}(T) \leqq \frac{n(h+K-1)}{2}\left(\frac{2^{hk}}{2^{hk}-1}\right) \qquad \text{all } h \geqq 1$$

We conclude this section by discussing systematic tree codes. The code is said to be systematic if $C_i^{(j)} = I_i^{(j)}$ for $1 \leqq j \leqq k$. Notice that in T_h, this means that $C_i^{(j)} = 0$ for $i \geqq h+1$, $1 \leqq j \leqq k$, so that $k(K-1)$ of the $n(h+K-1)$ code word positions are identically zero. This reduces the effective block length of T_h, and we obtain

THEOREM 2. If T is systematic

$$d_{\text{free}}(T) \leqq \frac{n(h+K-1) - k(K-1)}{2}\left(\frac{2^{hk}}{2^{hk}-1}\right)$$

$$\text{all } h \geqq 1$$

3. Improvement for Linear (Convolutional) Codes

If the code is linear, then the codes T_h are also, and instead of Plotkin's bound, we may use Griesmer's bound (Ref. 2), which improves Plotkin's slightly. Griesmer's bound says that if $d = d_0$ is the minimum distance of an (n, k) binary linear code, and if $d_i = [(d_{i-1}+1)/2]$, then $d_0 + d_1 + \cdots + d_{k-1} \leqq n$. For example, consider a rate 1/2, $K = 6$ convolutional code. Here T_3 is a (16,3) block code which has $d_{\min} \leqq 9$ by Plotkin's bound. But if $d_0 = 9$, then $d_1 = 5$, $d_2 = 3$ and $9 + 5 + 3 = 17 > 16$, so that no linear (16,3) code can have $d_{\min} = 9$. Thus a rate 1/2, $K = 6$ convolutional code must have $d_{\text{free}} \leqq 8$. Incidentally, this example suggests the interesting possibility that a nonlinear tree code might have $d_{\text{free}} = 9$, which would then be superior to any linear tree code.

4. A Short Table

We present in this section a short table of the upper bounds obtained by the arguments in this paper for nonsystematic tree codes (Table 2). The only rates considered are 1/2, 1/3, and 1/4. Column P is the Plotkin upper bound of *Subsection* 2, and column G lists the Griesmer upper bound of *Subsection* 3, whenever it is superior to the Plotkin Bound. The last column L gives the best d_{free} known to be achievable by a convolutional code.

References

1. Berlekamp, E. R., *Algebraic Coding Theory*, McGraw-Hill Publishing Co., New York, 1968.
2. Solomon, G., and Stiffler, J. J., "Algebraically Punctured Cyclic Codes," *Information and Control*, Vol. 8, pp. 170–179, 1965.

Table 2. Upper bounds

K	Rate 1/2			Rate 1/3			Rate 1/4		
	P	G	L	P	G	L	P	G	L
2	4		3	6		5	8		7
3	5		5	8		8	10		10
4	6		6	10		10	13		13
5	8		7	12		12	16		16
6	9	8	8	13		13	18		18
7	10		10	15		15	20		20
8	11		10	17	16	16	22		
9	12		12	18		18	25	24	24
10	13		12	20		20	27		
11	14		12	22			29		
12	16		13	24		22	32		
13	17	16		25	24		34	33	
14	18	17		27	26		36		
15	19	18		28			38		

E. Combinatorial Communications: Epsilon Entropy and Data Compression,

E. Posner and E. Rodemich

1. Introduction

The theory of efficient handling of data for transmission or storage is presently known as "data compression." As space exploration progresses to more and more distant